

# THE ORIGINS OF THE S CURVE IN BUSINESS FUNCTIONS

John Drummond, March 2012

The S Curve in BF was designed to spread the cost of a project in an S Curve. Without a formal definition of what an S Curve actually is, BF uses its own home-made S Curve.

The basic algorithm is based on a Sine wave. It was noticed that between 0 and  $\pi$  radians (0 and 180 degrees) the shape of a Sin wave is pretty much what you would expect the budget of a construction project to look like ie a bell shape.

But why do we call it an S Curve? Well the cumulative curve of the cost looks over S shape. What is the cumulative the cumulative curve from the sin to integrate the sin curve. If we do conveniently find that that the interal negative of the cosine curve:

$$\int \sin(x) dx = -\cos(x)$$

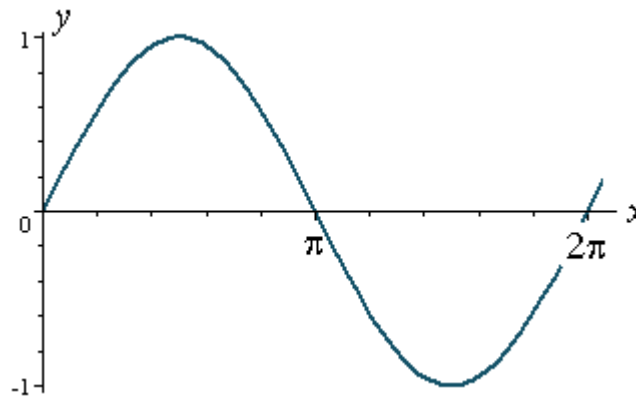


FIGURE 1 A SIN CURVE

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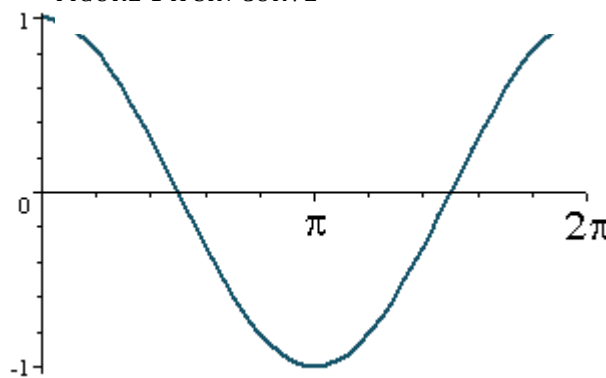


FIGURE 2 A COSINE CURVE

If you imagine that cosine curve upside down, between 0 and  $\pi$  you have a nice S curve. To make it fit more nicely with zero in the y axis and 1 at the top left hand corner of the 'S', we add 1 and half the result:

$$\frac{1}{2} (1 - \cos(x))$$

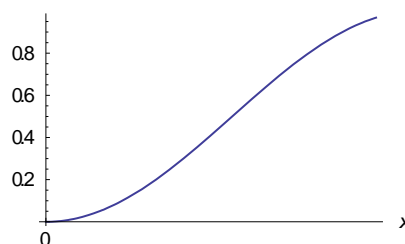


FIGURE 3 NEARLY THERE...

So far so good, it looks like an S curve, and the y axis goes as far as 1.0, which will be the total cost of the project. But the time on the x axis still isn't right. We want it to go to 1.0, where that indicates the total time of the project. In fact, we will define a "dimensionless time" as:

$$t = \frac{\text{Time From Start}}{\text{Total Project Time}}$$

At the moment, the S Curve plateaus out at  $x=\pi$ . That's because we measure the input to the cosine function as an angle, and therefore in radians, and  $\pi$  radians (or  $180^\circ$ ).

So our equation will work better if we use  $t$  instead of  $x$ , noting that:

$$x = \pi t$$

So:

$$\frac{1}{2} (1 - \cos(t \pi))$$

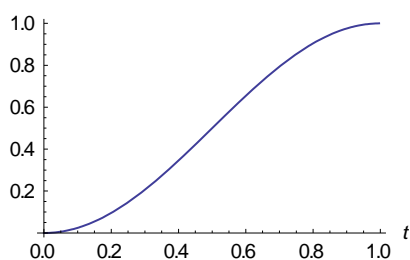


FIGURE 4 THE BASIC S CURVE

This curve has its uses: for any value of "dimensionless time" we can look up the "dimensionless cost", let's call it  $c$ , and from that work out the total cost incurred up until that time.

We could also use mathematical differentiation to work out the *rate* of production at any time:

$$\frac{d}{dt} \left( \frac{1}{2} (1 - \cos(t \pi)) \right) = \frac{1}{2} \pi \sin(\pi t)$$

If we plot out the equation for the rate:

$$\frac{1}{2} \pi \sin(\pi t)$$

We get the familiar bell curve of the rate of cost disbursement:

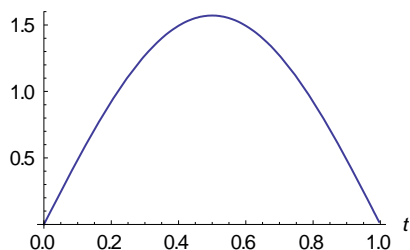


FIGURE 5 BELL CURVE OF THE INSTANTANEOUS RATE

But we only have one shape of curve. It would be nice to be able to 'tune' this curve according to how we think the actual curve might look.

Let's start by trying to skew the curve so that cost is either front or back loaded. The easiest way to do this is actually to adjust the time, I will use the delightful term "skewed time". If we have a variable  $s$  called skewness, we could skew the time simply by saying:

$$\text{Skewed Time } t_s = t^s$$

At high values of  $s$ , time will be substantially reduced, because the value of  $t$  is between 0 and 1. That will mean for lowish values of  $t$  the cost looked up will be much smaller, until  $t$  is close to 1 when it will quickly move to the value of 1 and the cost accrued will rapidly 'catch up'. Likewise, low values will frontload the cost.

So our S Curve becomes:

$$\frac{1 - \cos(\pi t_s)}{2}$$

Or expressed as  $t$  and making the skewed time more explicit:

$$\frac{1 - \cos(\pi t^s)}{2}$$

The other thing, apart from the Skewness, which we would like to change, would be the degree of "Peakness" of the bell curve, in other words how much cost happens in the middle of the cost disbursement period as opposed to the tails at either end.

The way this was approached is to say, OK, we have an S Curve, but we now want to be able to flatten it. In fact, the term we came up with "Peakness", should really be called "Flatness". Here's why.

Let us take the average of our S Curve and a completely uniform, even, spread of the cost over the period. The equation for the equal spread of cost is easy, its just  $t$ , ie the cumulative cost at any time is just equal to the dimensionless time. If we are first of all skewing our time, then the cumulative cost is just  $t_s$ .

Now let us take our "Peakness" variable (aka Flatness, apologies for the misnomer) and take the average of one S Curve and  $p/2$  uniform spreads. Why  $p/2$ ? It just works better with the algebra, and, provided we know what we are doing when we specify a value for  $p$ , its fine. So we take the average, noting that the denominator has to be the total number of curves.

$$\text{Average Curve} = \frac{1 \times S \text{ Curve} + \frac{p}{2} \times \text{Uniform Curves}}{1 + \frac{p}{2}}$$

Putting in the algebraic values for the curves and multiplying top and bottom by 2 we get:

$$\text{Cum Cost} = \frac{1 - \cos(\pi t^s) + p t^s}{2 + p} \quad \dots \text{Equation (1)}$$

The values of Peakness are between zero, which means it's the standard cosine curve, and infinity, a completely flat, uniform spread.

In BF, we mostly use the cumulative curve, to calculate the amount accrued to the end of the period and deduct from that the accrual to the start. But it would be nice to be able to get the instantaneous rate too, which means finding the derivative:

$$\frac{\partial}{\partial t} \left( \frac{1 - \cos(\pi t^s) + p t^s}{2 + p} \right) = \frac{s t^{s-1} (p + \pi \sin(\pi t^s))}{p + 2}$$

ie:

$$\text{Rate of Cost Disbursement} = \frac{s t^{s-1} (p + \pi \sin(\pi t^s))}{p + 2} \quad \dots \text{Equation (2)}$$

Equations (1) and (2) are the ones to use, but note that both time and the total amount are expressed as dimensionless amounts, so you need to calculate the dimensionless time as (years/months/whatever)/Total Time, and the output, dimensionless cost, must be multiplied by the Total Cost.

*Example:*

A project costs £100M and takes 5 years to build. It's a bit flatter than the standard cosine curve (Peakness=0.5), and bit skewed towards the back end (Skewness=1.5).

Find the amount of cost disbursed in Year 4 of the project and the instantaneous annual rate of cost disbursement half way through Year 4.

(Note that when using trigonometric functions, the equations assume an input in radians. Many calculators assume degrees whilst spreadsheets like Excel assume Radians.)

*Answer:*

$$\text{Cumulative Cost after 3 years} = \frac{1 - \cos\left(\pi\left(\frac{3}{5}\right)^{1.5}\right) + 0.5 \times \left(\frac{3}{5}\right)^{1.5}}{2 + 0.5} \times \text{£100M} = \text{£44.88M}$$

$$\text{Cumulative Cost after 4 years} = \frac{1 - \cos\left(\pi\left(\frac{4}{5}\right)^{1.5}\right) + 0.5 \times \left(\frac{4}{5}\right)^{1.5}}{2 + 0.5} \times \text{£100M} = \text{£79.37M}$$

$$\Rightarrow \text{Amount disbursed in Year 4} = 79.37 - 44.88 = \text{£34.49M}$$

$$\begin{aligned} \text{Rate of Cost Disbursement} &= \frac{1.5 \times \left(\frac{3.5}{5}\right)^{1.5-1} \left(0.5 + \pi \sin\left(\pi\left(\frac{3.5}{5}\right)^{1.5}\right)\right)}{0.5 + 2} \times \text{£100M} \\ &= \frac{1.25 \times 3.53}{2.5} \times \text{£100M} = \text{£176.5M Per 5 years} = \text{£35.3M per year} \end{aligned}$$